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Lattice conformal theories and their integrable perturbations

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Abstract

We study the lattice analogues of the Wess–Zumino–Witten (WZW) and Toda conformal field theories. We describe discrete versions of the Drinfel'd–Sokolov reduction and the Sugawara construction for the WZW model, and show how to formulate a perturbation theory in the chiral sector. We describe the spaces of integrals of motion of the perturbed theories. We interpret the perturbed WZW model in terms of NLS hierarchy and obtain an embedding of this model into the lattice KP hierarchy.

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1. Introduction

Studying discrete analogues of continuous classical and quantum two-dimensional integrable theories during the last two decades resulted in lots of new interesting mathematical constructions and helped in understanding physics of quantum problems, in which the regularization procedure was of great importance.

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A system with n degrees of freedom is integrable, if one can find n conserved quantities. Thus, studying the space of the integrals of motion of any system is one of the main steps in solving the theory. In a recent series of papers Feigin and Frenkel [28–30] proposed a new approach to study integrable systems. They have shown that integrals of motion of classical Toda field theories can be interpreted as cohomologies of certain complexes. Moreover, they have proved that these cohomologies allow for quantum deformation.

In the present paper we use Feigin–Frenkel method to calculate the integrals of motion of some discrete integrable theories. Special attention is paid to the lattice Wess–Zumino–Witten model (WZW) [1,2,23] and its integrable perturbations and lattice Toda theories.

Using cohomological technique, we show how the Drinfel'd–Sokolov [16] reduction can be derived on the lattice, starting from the analogue of the Wakimoto construction [47].

The paper is organized as follows. In Section 2 with a simple example we review the main ideas of Feigin–Frenkel approach. In Section 3 we remind the St.-Petersburg definition of lattice KM algebra, introduce convenient analogue of the Chevalley basis and describe the free fields representation of the lattice WZW model. In Section 4 we describe explicitly lattice Drinfel'd–Sokolov reduction and in Section 5 – lattice Sugawara construction. Then in Section 6 we study the perturbed lattice WZW model. For the sake of simplicity we restrict ourselves with $\widehat{sl}_2$ case. We describe lattice Maxwell–Bloch (MB) system, for which in continuous case integrals of motion were calculated recently [4]. We argue that this system can be treated as a proper lattice counterpart of the NLS hierarchy. In Section 7 we study the connection between lattice NLS and "universal" lattice KP hierarchies. We find that lattice NLS hierarchy provides special two-field realization of lattice KP. We also discuss lattice analogues of the affine Toda field theories and calculate their integrals of motion. We end up with some concluding remarks and review of unresolved questions.

2. Feigin–Frenkel approach

2.1. Continuous Toda theories

In this section we briefly review the main ideas of the cohomological approach following the papers [28–30,33].

The main object of study is a Hamiltonian formalism for the Toda field theories. It implies constructing a Hamiltonian space π and a Hamiltonian H such that the system of equations of motion can be rewritten in the Hamiltonian form:

$$\partial_\tau U = \{U, H\}_\pi . \tag{2.1}$$

Here $\{\,,\}_\pi$ denotes the Poisson structure on the space $\mathcal{F}(\pi)$ of functions on π. We are interested in finding the integrals of motion for Eq. (2.1). By definition, integral of motion I is an element of the space $\mathcal{F}(\pi)$ satisfying the equation

$$\{I, H\} = 0.$$

It is a conservation law of the Hamiltonian system, defined by Eq. (2.1).

This definition does not require H to be an element of $\mathcal{F}(\pi)$. The Poisson bracket with H must be a well-defined linear operator acting from $\mathcal{F}(\pi)$ to some other linear space. Given H, one can define the space of integrals of motion of the system (2.1) as the kernel of this linear operator. Moreover, if this operator preserves the Poisson structure, then the space of integrals of motion is itself a Poisson algebra.

For the Toda field theory one can choose as a Hamiltonian space, the space $\boldsymbol{Lh}$ of polynomial functions on a circle with coordinate x with values in the Cartan subalgebra $\boldsymbol{h}$ of semisimple algebra $\boldsymbol{g}$, or in other words, space of differential polynomials in the coordinates $u^i(x)$ of $\boldsymbol{u}(x) \in L\boldsymbol{g}$ with respect to the basis of the simple roots $\{\alpha_i\}$. Hereafter, we denote this space by π_0. As the space of functions, we take the space $\mathcal{F}_0$ of local functionals on π_0. Any such functional can be presented in the form

$$F[\boldsymbol{u}(x)] = \int P(\boldsymbol{u}(x), \partial_x \boldsymbol{u}(x), \ldots)\, \mathrm{d}x,$$

where $P \in \pi_0$. In addition, we can consider the spaces $\mathcal{F}_{\alpha_i}$, consisting of functionals of the form

$$\int P(\boldsymbol{u}(x), \partial_x \boldsymbol{u}(x), \ldots) \mathrm{e}^{\phi_i(x)}\, \mathrm{d}x,$$

where $\phi_i(x)$ are defined by $\partial_x \phi_i(x) = u^i(x)$. The Poisson bracket $\mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_0$ can be extended to a bilinear map $\mathcal{F}_0 \times \mathcal{F}_{\alpha_i} \to \mathcal{F}_{\alpha_i}$ [16]. This allows to write the Toda equations

$$\partial_\tau \partial_x \phi_i(x, \tau) = \sum_{\alpha_j - \text{simple}} \frac{(\alpha_i, \alpha_j)}{2} \mathrm{e}^{\phi_j(x,\tau)}, \qquad \alpha_i - \text{simple}$$

in the desired Hamiltonian form (2.1) as

$$\partial_\tau \boldsymbol{u}(x) = \{\boldsymbol{u}(x), H\}. \tag{2.2}$$

Here the Hamiltonian H is given by

$$H = \frac{1}{2} \sum_i Q_i, \quad Q_i = \int \mathrm{e}^{\phi_i(x)}\, \mathrm{d}x. \tag{2.3}$$

It is an element of $\bigoplus_i \mathcal{F}_{\alpha_i}$, and the Poisson bracket with H is a well-defined linear operator, acting from $\mathcal{F}_0$ to $\bigoplus_i \mathcal{F}_{\alpha_i}$. Thus, the space of integrals of motion of the Toda equation (2.2) can be defined as the kernel of the operator $ad_H \equiv \{\bullet, H\}$, or as the intersection of the kernels of the operators $ad_{Q_i}\colon \mathcal{F}_0 \to \mathcal{F}_{\alpha_i}$. These operators preserve the Poisson structure on $\mathcal{F}_0$, and hence the space of integrals of motion is a Poisson subalgebra of $\mathcal{F}_0$.

To illustrate the general scheme, consider first the Liouville theory. The space π_0 is defined as the space of differential polynomials in variable $\partial_x \phi$, Poisson bracket being given by [40]

$$\{\partial_x \phi(x), \partial_y \phi(y)\} = \delta'(x - y).$$

Hamiltonian of the theory is $H = \bar{Q} = \int \mathrm{e}^{\phi(x)}\, \mathrm{d}x \in \mathcal{F}_1$. The goal is to calculate the space of local integrals of motion $I_0(sl_2)$ defined as the kernel of the linear operator $ad_{\bar{Q}} \equiv \{\bullet, \bar{Q}\}$. Due to Jacobi identity, it is closed with respect to Poisson bracket.

Feigin and Frenkel have proved in [29] that $I_0(sl_2)$ consists of local functionals, defined by differential polynomials, depending on a single variable

$$T(x) = \tfrac{1}{2}(\partial_x\phi)^2 - \partial_x^2\phi.$$

When stated, this result is obvious, as it is easy to check, that $T(x)$ itself commutes with $\bar{Q}$, and hence so does any differential polynomial of T. $T(x)$ realize the classical Virasoro algebra

$$\{T(x), T(y)\} = (\partial_x^3 + 2T\partial_x^2 + \partial_x T\partial_x)\delta(x-y).$$

For the case of general semisimple Lie algebra the Poisson algebra $I_0(g)$ of local integrals of motion has been shown to coincide with the Adler-Gel'fand–Dickey algebra, otherwise called classical W-algebra.

The Drinfel'd–Sokolov reduction [16] allows one to obtain this algebra as the zeroth cohomology of the corresponding classical BRST complex. In Section 4 we obtain similar results for the lattice model.

In the case of the *affine* Toda field theory the Hamiltonian is given by

$$\tilde{H} = H + Q_0, \tag{2.4}$$

where $Q_0 = \int \mathrm{d}x\, \mathrm{e}^{-\sum_i \phi_i}$. Referring the reader to the original papers for the details [28,29], we just state the answer here.

The space $I_0(g)$ of local integrals of motion of the Toda theory, associated to an affine algebra g, is linearly generated by mutually commuting local functionals of degrees equal to the exponents of g modulo the Coxeter number.

2.2. Lattice Toda theories

In this section we consider the lattice analogue of the contsruction described in the previous section. We consider the lattice Liouville model [5,6,20,25,45], following mainly [25]. The space π_0 in that case is the space of finite-difference polynomial functions on a discrete circle with coordinate $n = 0, 1, 2, \ldots, N$ with values in the lattice abelian current algebra [22,46], i.e. the space of finite-difference polynomials in the variable u_n, with the Poisson bracket defined as

$$\{u_n, u_m\} = u_n u_m(\delta_{n,m+1} - \delta_{m,n+1}). \tag{2.5}$$

The space of local functionals $\mathcal{F}_0$ is defined through the summation map $\Sigma : \pi_0 \to \mathcal{F}_0$. Any such functional can be presented in the form

$$F[u_n] = \sum_n P(u_n, u_n - u_{n+1}, \ldots),$$

where $P \in \pi_0$. There is also an appropriate lattice counterpart for the field $\mathrm{e}^{\phi(x)}$, which we denote a_n. Its relation to the current u_n is expressed by the formula

$$u_n = a_n a_{n+1}^{-1}.$$

The Poisson bracket for a_n has the form

$$\{a_n, a_m\} = a_n a_m \mathrm{sign}(n - m). \tag{2.6}$$

It is easy to see that (2.5) and (2.6) are consistent.

Lattice Liouville Hamiltonian is defined as $H \equiv \bar{Q} = \sum_n a_n$. The space of local integrals of motion $I_0^L(sl_2)$ (L for "lattice") is defined as the kernel of the linear operator $ad_{\bar{Q}}$, acting from $\mathcal{F}_0$ to $\mathcal{F}_1$, where $\mathcal{F}_1$ is defined as space of local functionals of the form

$$F_1 = \sum_n P(u_n, u_n - u_{n+1}, \ldots)\, a_n,$$

where $P \in \pi_0$.

It has been shown in [25] that $I_0^L(sl_2)$ consists of local functionals, defined by finite-difference polynomials, depending on a single variable

$$A_n = \frac{u_{n+1}}{(1 + u_n)(1 + u_{n+1})}. \tag{2.7}$$

This formula first appeared in the paper [20], where it was considered as a lattice analogue of the Miura transformation for the classical Virasoro algebra. A_n's form the classical lattice Virasoro–Faddeev–Takhtadjan–Volkov (FTV) algebra

$$\begin{aligned} \{A_n, A_{n+1}\} &= -A_n A_{n+1}(1 - A_n - A_{n+1}), \\ \{A_n, A_{n+2}\} &= -A_n A_{n+1} A_{n+2}. \end{aligned} \tag{2.8}$$

As in the continuous case, for the lattice Toda theory associated to the semisimple Lie algebra $\boldsymbol{g}$ one can prove that the space of local integrals of motion is a Poisson algebra, generated by r elements, where $r = \mathrm{rank}(\boldsymbol{g})$. This algebra has a natural interpretation as a lattice analogue of the Gelfand–Dickey algebra (see Section 7 for explicit formulae).

3. Lattice Kac–Moody algebra and WZW model

3.1. *Lattice Kac–Moody algebra – St.-Petersburg definition*

In this section we define the lattice Kac–Moody algebra (LKMA) following the papers of the St.-Petersburg group [1,2,43]. The following exchange relations were proposed for the quantum lattice L-operator (discrete analogue of the Kac–Moody current):

$$\begin{aligned} J(n)_1 J(n)_2 &= R^+ J(n)_2 J(n)_1 R^-, \\ J(n+1)_1 R^- J(n)_2 &= J(n)_2 J(n+1)_1. \end{aligned} \tag{3.1}$$

The standard notations $A_1 \equiv A \otimes 1$, $A_2 \equiv 1 \otimes A$ are used. R^+ and $R^- = P(R^+)^{-1}P$, where P is the permutation operator, satisfy quantum Yang–Baxter equation without spectral parameter

$$R_{12}^{\pm} R_{13}^{\pm} R_{23}^{\pm} = R_{23}^{\pm} R_{13}^{\pm} R_{12}^{\pm}.$$

For the sl_2 case these matrices have the following form

$$R^+ = q^{1/2}\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \quad R^- = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

For further convenience we define an analogue of the Chevalley basis for the LKMA. Instead of the matrix form of LKMA (3.1) with current matrix

$$J(n) = \begin{pmatrix} J(n)_{11} & J(n)_{12} \\ J(n)_{21} & J(n)_{22} \end{pmatrix}$$

with "sl_2-constraint":

$$J(n)_{11}J(n)_{22} - q^{-1}J(n)_{21}J(n)_{12} = q^{1/2}$$

we will use the variables

$$e_n = J_n^{12}J_n^{22}, \qquad f_n = J_n^{21}(J_n^{22})^{-1}, \qquad h_n = (J_n^{22})^2 \tag{3.2}$$

with exchange relations

$$\begin{aligned} h_n h_{n+1} &= q^{-1}h_{n+1}h_n, & & \\ h_n e_n &= q^{-1}e_n h_n, & h_n f_n &= q f_n h_n, \\ h_n e_{n+1} &= q^{-1}e_{n+1}h_n, & h_n f_{n+1} &= q f_{n+1}h_n, \\ e_n f_n &= q f_n e_n + q - 1, & [e_n, f_{n+1}] &= -(q-1)q^{-1/2}h_n. \end{aligned} \tag{3.3}$$

In the quasi-classical limit ($q \to 1$, with the appropriate scaling of the Poisson brackets) one obtains:

$$\begin{aligned} \{h_n, h_{n+1}\} &= -h_n h_{n+1}, & & \\ \{h_n, e_n\} &= -h_n e_n, & \{h_n, f_n\} &= h_n f_n, \\ \{h_n, e_{n+1}\} &= -h_n e_{n+1}, & \{h_n, f_{n+1}\} &= h_n f_{n+1}, \\ \{e_n, f_n\} &= 1 + e_n f_n, & \{e_n, f_{n+1}\} &= -h_n. \end{aligned} \tag{3.4}$$

3.2. Lattice Wakimoto construction

In this section we describe the realization of LKMA in terms of free fields. In continuous case there exists such a Fock space realization of $\hat{\mathcal{G}}$. It was first obtained for $\widehat{sl(2)}_1$ by Wakimoto [47] and later generalized for $\widehat{sl(n)}_k$ by Feigin and Frenkel [27]. More of an algebraic approach to the same problem was taken by Bouwknegt et al. [12]. It has been shown that the Fock-space modules for $\widehat{sl(n)}$ form the complex. The intertwining operators that build the complex, realize the action of the $U_q(n_+)$ on the Fock space. Explicit formulae for the intertwining operators (also called Screening Charges (SC)) in terms of free fields can also be found in [12].

Below, following the main steps of [12], we present an analogue of the Fock-space realization of the LKMA. The Fock space of local fields on the lattice is defined as a space of finite-difference zero-degree polynomials of the following variables:

– lattice vertex operators a_n^i, $i = 1, 2, \ldots, \#$(simple roots) with exchange relations

$$\begin{aligned} a_n^i a_{n+m}^j &= q^{A_{ij}/2} a_{n+m}^j a_n^i, \quad \text{for } m > 0, \\ a_n^i a_n^{i+1} &= q^{1/2} a_n^{i+1} a_n^i, \\ a_n^i a_n^j &= a_n^j a_n^i, \quad \text{when } | i - j | \geq 2, \\ deg(a_n^i) &= \alpha_i, \end{aligned} \tag{3.5}$$

where A_{ij} is the Cartan matrix of $sl(n)$. Zero-degree elements (analogues of the Cartan currents) are defined as $p_n^i \equiv a_n^i (a_{n+1}^i)^{-1}$;

– lattice $\beta - \gamma$ systems, labelled by the positive roots, with exchange relations

$$\begin{aligned} &B_n^\alpha \Gamma_n^\alpha = q \Gamma_n^\alpha B_n^\alpha + q - 1 \equiv q \xi_n^\alpha - 1, \\ &deg(B_n^\alpha) = deg(\Gamma_n^\alpha) = 0, \end{aligned}$$

where $\xi_n^\alpha = 1 + \Gamma_n^\alpha B_n^\alpha$.[4]

We denote the space of finite-difference poynomials of a_n^i, B_n^α, Γ_n^α by π_α. Quite similar to the case of Toda theories, considered previously, and in analogy with continuous case [12], one can define the screening charges Q_α acting from π_0 to π_α. Using the language of Section 2, the space of local integrals of motion will coincide with LKMA.

Here we present explicit calculations for the sl_2-case. Relations (3.5) and (3.6) amount to:

$$a_n a_{n+m} = q^{-1} a_{n+m} a_n \qquad \beta_n \gamma_n = q \gamma_n \beta_n + q - 1 \equiv q \xi_n - 1. \tag{3.6}$$

Screening operators are given by the formulae

$$Q_1 \equiv Q_{\alpha_1} = \sum_n a_n \beta_n, \qquad Q_0 \equiv Q_{\alpha_0} = \sum_n a_n^{-1} \gamma_n,$$

$\alpha_1 = (1, -1)$ is the simple root of $sl(2)$, $\alpha_0 = -\alpha_1$ is the affine root of $\widehat{sl(2)}$. Q_1 is the single generator of $U_q n_+(sl(2))$, and together with Q_0 they form Chevalley basis of $U_q n_+(\widehat{sl(2)})$. One finds by direct computation that the combinations

$$e_n = \beta_n, \qquad f_n = \gamma_n - q^{-1/2} \gamma_{n-1} \xi_n p_{n-1}, \qquad h_n = p_n \xi_n \xi_{n+1}, \tag{3.7}$$

obey the LKMA in Chevalley basis (3.4). In continuous limit formulae (3.7) coincide with the quasi-classical limit of Wakimoto construction [27,47]:

$$\begin{aligned} &f(z) = - : \gamma\gamma\beta : (z) - \sqrt{2(k+2)} \gamma(z) H(z) - k \partial \gamma(z), \\ &h(z) = 2 : \gamma\beta : (z) + \sqrt{2(k+2)} H(z), \\ &e(z) = \beta(z). \end{aligned}$$

For the first time Wakimoto bosonization on the lattice was proposed in the paper [23], where the authors using the similar system of free fields, constructed the lattice analogue of

[4] The exchange relations between ξ and original variables are $B\xi = q \xi B$ and $\Gamma \xi = q^{-1} \xi \Gamma$.

Bernard–Felder cohomology. However, their explicit realization of the LKMA is different from the one given by (3.7).

4. Lattice Drinfel'd–Sokolov reduction

The Drinfel'd–Sokolov (DS) reduction is known to be the most powerful method of constructing W-algebras in continuous field theory. In this section we show, how a version of the DS reduction can be realized on the lattice, giving rise to lattice W-algebras. Surprisingly, the procedure we are going to present is not much different from its continuous analogue. In the rest of this section we will be closely following the excellent review paper [13], where also the original references on DS reduction can be found.

In this method one starts with an affine Lie algebra $\hat{g}_k$ (in our case with a LKMA), an affine subalgebra $\hat{g}'$ and reduces it by imposing some first order constraints $g \sim \chi(g)$ on the generators $g \in \hat{g}'$, where $\chi(g)$ is some one-dimensional representation of $\hat{g}'$. On the classical level, which is the only one we consider in this paper, this procedure gives Gel'fand-Dickey algebras $W[\hat{g}, k]$ (lattice W-algebras). Let us choose $\hat{g}'$ to be the upper Borel part of $\hat{g}_k$.

A set of constraints can be imposed by means of the appropriate BRST procedure, and the reduced algebra is defined as the cohomology of the BRST operator.

In this paper we consider only the case of lattice sl_2. Denote the generators of this algebra through e_n, f_n, h_n, as in Section 3.2. To impose the constraint, we notice that the upper Borel part of lattice sl_2 consists of a single element e_n, so the constraint is simply $e_n = 1$. To implement the BRST procedure, we need to introduce two ghost fields b_n, c_n satisfying the Poisson algebra

$$\{b_n, c_m\} = \delta_{nm}$$

and anticommutation relations:

$$b_n b_m + b_m b_n = c_n c_m + c_m c_n = b_n c_m + c_m b_n = 0.$$

By analogy with the continuous case described in [13], we define the BRST operator as

$$Q = Q_0 + Q_1, \qquad Q_0 = \sum_n c_n e_n, \qquad Q_1 = \sum_n c_n.$$

It easy to check that Q defines a double complex, i.e. $Q^2 = Q_0^2 = Q_1^2 = Q_0 Q_1 + Q_1 Q_0 = 0$. To calculate the cohomologies of this double complex, we use the spectral sequence technique in the same way it is used in the usual DS reduction. The spectral sequence terminates on the second step, so that

$$H_Q(*) \simeq H_{Q_1}(H_{Q_0}(*)).$$

The Q_0-cohomologies are generated by the fields c_n and $\tilde{h}_n = h_n(1 - b_n c_n)(1 - b_{n+1} c_{n+1})$. The Poisson algebra of c_n and $\tilde{h}_n$ is not free, however, in the Q_0-cohomology space: there exists a field $N_n = c_n \tilde{h}_n - c_{n+1}$, satisfying $\{Q_0, N_n\} = 0$. The Poisson algebra factorized by the condition $N_n \sim 0$ is isomorphic to the Poisson algebra for lattice "vertex operator" a_n and lattice $U(1)$-current $p_n = a_n^{-1}a_{n+1}$ (see Section 3.2 for the definitions). Thus, calculation of the Q_1- cohomologies of complex $Q_0(*)$ reduces to Feigin's construction of the lattice W-algebra [25] (lattice Virasoro or Faddeev–Takhtadjan–Volkov (FTV) algebra in our $\widehat{sl}_2$-case). We have to find the kernel of the screening operator

$$Q_1 = \sum_n a_n.$$

The answer can be found in [25]:

$$\tilde{A}_n = \frac{1}{(1+\tilde{h}_n)(1+\tilde{h}_{n+1}^{-1})}$$

defines the appropriate cohomology class

$$\{Q_1, \tilde{A}_n\} = -\hat{N}_n - \hat{N}_{n+1} \sim 0,$$

where we defined

$$\hat{N}_n = \frac{N_n}{1+\tilde{h}_n} \sim 0.$$

It is easy to check that $\tilde{A}_n$ forms FTV algebra

$$\{\tilde{A}_n, \tilde{A}_{n+2}\} = \tilde{A}_n\tilde{A}_{n+1}\tilde{A}_{n+2}, \qquad \{\tilde{A}_n, \tilde{A}_{n+1}\} = \tilde{A}_n\tilde{A}_{n+1}(-1+\tilde{A}_n+\tilde{A}_{n+1}). \tag{4.1}$$

It is also easy to verify that N_n (and hence $\hat{N}_n$) form an ideal in the Poisson algebra of c_n and $\tilde{h}_n$:

$$\{c_{n+1}, N_n\} = -c_nN_n, \qquad \{\tilde{h}_n, N_n\} = \tilde{h}_nN_n,$$
$$\{\tilde{h}_{n-1}, N_n\} = \tilde{h}_{n-1}N_n, \qquad \{N_n, N_n\} = -(c_{n-1}+c_n)N_n.$$

Now we want to find another cohomological class B_n such that $\{Q, B_n\} = 0$, without any null-fields on the right-hand side. In other words, B_n should represent the same cohomology class of a double complex as A_n does, but on the original phase space, not factorized over the condition $N_n \sim 0$. In Section 5 we will consider the Sugawara construction as an example of such a class. Here we explain how to organize the "improvement" process. The idea of construction of the class B_n is to find such corrections to $\ln \tilde{A}_n$ which kill $\hat{N}_n$ terms. This can be done with the help of a staircase sequence in the double complex. Consider the following sequence:

$$
\begin{tikzcd}
0 & & & \\
\ln\tilde{A}_n \arrow[u, "\{Q_0,\bullet\}"'] \arrow[r, "\{Q_1,\bullet\}"'] & -(\tilde{N}_n+\tilde{N}_{n+1}) & & \\
& -(\phi_n+\phi_{n+1}) \arrow[u, "\{Q_0,\bullet\}"'] \arrow[r, "\{Q_1,\bullet\}"'] & \phi_n\tilde{N}_n+\phi_{n+1}\tilde{N}_{n+1} & \\
& & \tfrac{1}{2}\phi_n^2+\tfrac{1}{2}\phi_{n+1}^2 \arrow[u] \arrow[r] & \cdots,
\end{tikzcd}
$$

where $\phi_n = (\tilde{f}_{n+1})/(1+\tilde{h}_n)$. After the summation of this (infinite) process one obtains

$$\ln B_n = \ln\tilde{A}_n + \sum_{m=1}^{\infty}\frac{\phi_n^m+\phi_{n+1}^m}{m} \quad \text{or} \quad B_n = \frac{\tilde{A}_n}{(1-\phi_n)(1-\phi_{n+1})}.$$

It is easy to check that B_n satisfies both desired propeties: it commutes with $Q = Q_0 + Q_1$ and obeys the same FTV algebra (4.1) as $\tilde{A}_n$ does.

5. Lattice Sugawara construction

In this section we are going to discuss the analogue of the Sugawara construction on the lattice. The question of what object is to be considered an analogue of the Sugawara element is rather ambiguous. In continuous case the Sugawara stress–energy tensor $T(z) = \sum_a (J^aJ^a)(z)$ possesses a number of peculiar properties, and it is not obvious which can be taken as the definition. Before proceeding with the calculations, let us make one comment concerning the classical case. In continuum, the Sugawara element satisfies the second Gel'fand–Dickey Poisson algebra with *zero* central charge.[5] On the other hand the continuous limit of FTV algebra (4.1) reproduces the classical Virasoro algebra with necessarily *non-zero* central term. This makes one believe that the generator of the FTV algebra should contain some twisting part in the continuous limit independent of the elements of the underlying algebra it is built of. In the course of DS reduction such a twisted energy–momentum appears naturally and is given by

$$T(z) = \frac{1}{2(h+2)} : J^+J^- + J^-J^+ + \frac{1}{2}J^0J^0 : + \frac{1}{2}\partial J^0 + : \partial b c : . \tag{5.1}$$

[5] In quantum case, however, the central charge becomes non-zero due to quantum corrections.

Below we construct such a class $A_n^{\rm sug}$ which coincides in the continuous limit with (5.1).

For this purpose we start with

$$A_n = \frac{1}{(1+h_n)(1+h_{n+1}^{-1})}$$

and after summation of a certain staircase process (slightly more complicated than the one constructed above) obtain the desired class

$$A_n^{\rm sug} = \frac{1}{(1+h_n+x_n)(1+h_{n+1}^{-1}+y_n)}, \tag{5.2}$$

where x_n and y_n are net corrections (after summation of a staircase process). The explicit form of x_n and y_n is as follows:

$$x_{2n} = h_{2n}M_{2n-1}^1 \equiv h_{2n}\frac{e_{2n-1}f_{2n}}{h_{2n+1}}, \qquad y_{2n} = h_{2n+1}^{-1}M_{2n+1}^0 \equiv h_{2n+1}^{-1}e_{2n+1}f_{2n+1},$$

$$x_{2n+1} = M_{2n+1}^0 \equiv e_{2n+1}f_{2n+1}, \qquad y_{2n+1} = M_{2n+1}^1 \equiv \frac{e_{2n+1}f_{2n+2}}{h_{2n+1}}.$$

Replacing $M_{2n-1}^1 \to M_{2n}^0$ and $M_{2n+1}^0 \to M_{2n+1}^1$ in the above gives another copy of the FTV algebra.

It is easy to see that the field (5.2) obeys FTV algebra (4.1) and in the continuous limit (in the leading non-trivial order of a lattice spacing Δ) reduces to the classical limit of (5.1). After suppressing ghost fields ($b_n = c_n = 0$) one obtains twisted lattice Sugawara element. Naturally, $A_n^{\rm sug}[b_n = c_n = 0]$ obeys the same FTV algebra.

6. Perturbed lattice WZW model

6.1. Formulation of the model

In this section we describe the construction, to which we refer to as "lattice perturbed WZW model" having in mind the parallelism with a continuous case [4]. As in Section 3 we will not construct any Lagrangian perturbation theory, but rather consider Hamiltonian perturbation.

Below we present explicit calculations for the sl_2-case. Consider quasi-classical limit of Wakimoto lattice fields (3.6) a_n, β_n, γ_n, with following commutation relations:

$$\{a_n, a_{n+m}\} = a_n a_{n+m}, \qquad \{\gamma_n, \beta_n\} = \gamma_n\beta_n + 1. \tag{6.1}$$

Introduce the Hamiltonian of the lattice perturbed WZW model

$$H = \sum_n a_n\beta_n + \sum_n a_n^{-1}\gamma_n. \tag{6.2}$$

The first part of it

$$Q_1 = \sum_n a_n\beta_n$$

commutes with quasi-classical e_n, f_n, h_n currents, constructed in quantum case from elements a_n, β_n, γ_n by formulae (3.7).

The second part of Hamiltonian

$$Q_0 = \sum_n a_n^{-1} \gamma_n$$

can be treated as perturbation.

Consider the grading of $\beta - \gamma - a$ system

$$deg\, a_n = 1, \qquad deg\, a_n^{-1} = -1, \qquad deg\, \beta_n = deg\, \gamma_n = 0.$$

According to these rules we have $deg\, Q_0 = -1, deg\, Q_1 = 1$.

Let us introduce the adjoint action as improved Poisson brackets

$$ad_A B := \{A, B\} - (degA\; degB)AB.$$

With respect to the adjoint action operators Q_1 and Q_0 satisfy Serre relations for the nilpotent part of $\hat{sl}_2$ algebra

$$ad^3_{Q_0} Q_1 = ad^3_{Q_1} Q_0 = 0.$$

We consider the dynamical system with phase space being that of zero-degree polynomials lattice fields constructed from lattice $(\beta - \gamma - a)$-system and the Hamiltonian (6.2).

Our purpose now is to prove the integrability of this system and calculate the integrals of motion (IM), in analogy with the continuous case, considered in [4]. We will also give an interpretation of the model in terms of lattice analogues of the NLS hierarchy:

Let us start from an observation that the field h_n is a "zero mode" because it is conserved under the system evolution:

$$\dot{h}_n \;=\; ad_H(h_n) \;=\; 0.$$

This implies that it is necessary to reduce our dynamical system and exclude the field h_n. We introduce new lattice fields

$$x_n = \beta_n a_n, \qquad y_n = \gamma_n a_n^{-1}.$$

Corresponding Dirac brackets for these fields (up to a sign change) are:

$$\begin{aligned} \{x_n, x_m\}_D &= -\mathrm{sign}(n-m) x_n x_m, \\ \{y_n, y_m\}_D &= -\mathrm{sign}(n-m) y_n y_m, \\ \{x_n, y_m\}_D &= \mathrm{sign}(n-m) x_n y_m - \delta_{nm}(1 + x_n y_m). \end{aligned} \tag{6.3}$$

In these variables Hamiltonian has the form

$$H = \sum_n (x_n + y_n).$$

Poisson algebra (6.3) strongly reminds that of from the Feigin–Enriquez model (FE) [26]. The only difference is the δ_{nm} term in $x - y$ sector. This central term changes IM and dynamics drastically. Nevertheless, cohomological structure of the space of IM appears to be rigid with respect to such a deformation of Poisson brackets, as we are going to show now.

Consider the following one-parameter deformation of the Poisson structure of FE model:

$$\begin{aligned}\{x_n, x_m\}_D &= -\text{sign}(n-m)x_n x_m,\\ \{y_n, y_m\}_D &= -\text{sign}(n-m)y_n y_m,\\ \{x_n, y_m\}_D &= \text{sign}(n-m)x_n y_m - \delta_{nm}(\lambda + x_n y_m)\end{aligned} \tag{6.4}$$

with Hamiltonian

$$H = Q_+ + Q_-,$$

where $Q_+ = \sum_n x_n$ and $Q_- = \sum_n y_n$. For the lattice variables x_n and y_n we have

$$deg\, x_n = 1, \qquad deg\ y_n = -1.$$

For $\lambda = 0$ we obtain FE model and for $\lambda = 1$ we come to our initial algebra (6.3) corresponding to perturbed lattice WZW (or lattice NLS) model. It should be mentioned that all the Poisson algebras $\mathcal{A}_\lambda$ defined by bracket (6.4) are pairwise isomorphic for $\lambda \in (0, \infty)$.

In the paper [26] IM for the system (6.4) for $\lambda = 0$ have been expressed in terms of cohomology classes. Standard arguments give that the ring of cohomologies does not change under the infinitesimal variation of basic algebraic structure (6.4) on the phase space. The isomorphism of algebras $\mathcal{A}_{\lambda\neq 0}$ allows us to replace an infinitesimal deformation by the finite one. Thus, the rings of cohomologies for FE model and perturbed lattice WZW model are the same.

6.2. *Interpretation of the model in terms of the NLS hierarchy*

Before the systematic study of IM we give a brief description of our model in terms of lattice analogue of the NLS hierarchy. Let us first change the notations from x_n, y_n to the standard ones adopted in the theory of NLS equation: $x_n \equiv \psi_n$, $y_n \equiv \bar{\psi}_n$. We find that Eqs. (6.3) are exactly the first Poisson structure of the lattice NLS hierarchy:

$$\begin{aligned}\{\psi_n, \psi_m\} &= -\text{sign}(n-m)\psi_n\psi_m,\\ \{\bar{\psi}_n, \bar{\psi}_m\} &= -\text{sign}(n-m)\bar{\psi}_n\bar{\psi}_m,\\ \{\psi_n, \bar{\psi}_m\} &= -\text{sign}(n-m)\psi_n\bar{\psi}_m - (1+\psi_n\bar{\psi}_n)\delta_{n,m} + \delta_{m,n+1}.\end{aligned}$$

This bracket can be represented as a sum of two compatible Poisson structures, $\{\ ,\ \} = \{\ ,\ \}_1 + \{\ ,\ \}_0$, where $\{\ ,\ \}_1$ is defined as

$$\begin{aligned}\{\psi_n, \psi_m\}_1 &= -\text{sign}(n-m)\psi_n\psi_m,\\ \{\bar{\psi}_n, \bar{\psi}_m\}_1 &= -\text{sign}(n-m)\bar{\psi}_n\bar{\psi}_m,\\ \{\psi_n, \bar{\psi}_m\}_1 &= \text{sign}(n-m)\psi_n\bar{\psi}_m - \delta_{nm}(1+\psi_n\bar{\psi}_m),\end{aligned}$$

and $\{\ ,\ \}_0$ is defined as

$$\{\psi_n, \bar{\psi}_{n+1}\}_0 = 1.$$

One can easily find the first two integrals of the hierarchy

$$I_0 = \sum_n ln\,(1 + \psi_n \bar{\psi}_n), \qquad I_1 = \sum_n (\psi_n \bar{\psi}_{n+1}).$$

Two compatible brackets and two integrals of motion define a bi-Hamiltonian system, and hence, an infinite family of conservation laws. In continuous limit the few first IMs become

$$\begin{aligned} &I_0 \to N \text{ (number of particles)},\\ &I_1 - I_0 \to P \text{ (momentum)},\\ &I_2 - 2I_1 + I_0 \to \text{ (NLS Hamiltonian)}, \end{aligned}$$

where

$$I_2 = \sum_n \left(\frac{\psi_n^2 \bar{\psi}_{n+1}^2}{2} - \psi_n \bar{\psi}_{n+2} \right).$$

7. Embedding of the lattice NLS hierarchy into the lattice KP hierarchy

7.1. Lattice Toda theories and lattice analogue of the non-linear W_∞ algebra

We briefly remind here, how Feigin's construction of lattice W_N-algebra can be extended to the case of $N = \infty$ [8]. We consider here only the classical case.

Consider the set of lattice variables $\{a_n^j\}_{j=1}^N$ with the Poisson structure

$$\begin{aligned} &\{a_n^i, a_m^i\} = \mathrm{sign}(m-n) a_n^i a_m^i,\\ &\{a_n^i, a_n^{i+1}\} = -\tfrac{1}{2} a_n^i a_n^{i+1}, \qquad \{a_n^i, a_m^{i+1}\} = -\tfrac{1}{2}\mathrm{sign}(m-n) a_n^i a_m^{i+1}. \end{aligned} \tag{7.1}$$

Following the general scheme, we define the gradation on the phase space:

$$deg(a_n^i) = 1, \qquad deg((a_n^i)^{-1}) = -1 \quad i = 1, \ldots, N-1.$$

Let Π_n be the space of the finite-difference polynomials of degree n. The Hamiltonian of the lattice Toda theory associated with the finite-dimensional Lie algebra $sl(n)$ is given by

$$H_{sl_n\,Toda} = \sum_{i=1}^{n-1} Q_i, \tag{7.2}$$

where $Q_i = \sum_n a_n^i$ are the corresponding *screening charges* (SC). Through the tedious but straightforward calculation one can see that Q_i satisfy Serre's relations in $n_+(sl(n))$. The space of the integrals of motion is given by the intersection of the kernels

$$I_{lattice}(g) = \bigcap_{i=1}^{N-1} Ker(ad_{Q_i}) \cap \Pi_0. \tag{7.3}$$

It forms a Poisson algebra which can be viewed as a proper lattice analogue of the Adler–Gel'fand–Dickey, or W_N, algebra. We will denote this algebra as LW_N. In the papers [7,8,42] explicit calculations have been done for the $sl(3)$ case. In general, it was shown to be spanned by $N-1$ generators $L_n \equiv W_n^{(2)}, W_n^{(3)}, \dots, W_n^{(N)}$. Inductive limit $N \to \infty$ gives the lattice analogue of the classical non-linear W_∞-algebra.

To construct the generators $W_n^{(i)}$ we will use more convenient variables than a_n^i. First of all, we exclude the non-zero degree components of the phase space by using as the basic variables the lattice analogues of the Cartan currents of sl_N, associated with simple roots $\alpha_1, \alpha_2, \dots$:

$$p_n^i = a_n^i (a_{n+1}^i)^{-1}.$$

Calculations with these variables turned out to be rather tedious [7], so this time we choose another basis in the root system of sl_N and use Weyl chamber generators α_1, $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3, \dots$:

$$\begin{aligned} k_n^1 &= p_n^1, \\ k_n^2 &= p_n^1 p_n^2, \\ k_n^3 &= p_n^1 p_n^2 p_n^3, \\ &\vdots \\ k_n^{N-1} &= p_n^1 p_n^2 \cdots p_n^{N-1}. \end{aligned} \tag{7.4}$$

The following combinations turn out to be the best for our purposes:

$$\begin{aligned} \alpha_n^1 &= \sum_{i=1}^{N-1} k_n^i, \\ \alpha_n^2 &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} k_n^i k_{n+1}^j, \\ &\vdots \\ \alpha_n^p &= \sum_{\{u(l)\}} \prod_{j=1}^{p-1} k_{n+j}^{u(j)}, \end{aligned}$$

where the summation goes over all possible sets $\{u(l)\}$ such that $N-1 \geq u(l+1) > u(l) \geq 1$. They form a quadratic Poisson algebra

$$\{\alpha_n^p, \alpha_{n+m}^q\}_1 = \theta_m^{pq}(-\alpha_n^p \alpha_{n+m}^q + \alpha_n^{q+m} \alpha_{n+m}^{p-m}), \tag{7.5}$$

where $\theta_m^{pq} = \theta(p-m)\theta(q-p+m-1)$ and $\theta(x)$ is a step function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The index 1 of the bracket indicates that this is the analogue of *first* Gel'fand–Dickey (GD) structure for the N-KdV hierarchy. Generators of LW_N form the analogue of the *second* GD structure. It turns out that quite parallel to the continuous case there are Miura transformation, relating the α-fields (7.5) with the generators of LW_N. Direct calculation (the most convincing method of proof) shows that the following fields commute with all screening operators ($i = 2, 3, \ldots, N$):

$$\begin{aligned} W^{(i)}{}_n &= \frac{\alpha^{i-1}_{n+1} + \alpha^i_n}{(1+\alpha^1_n)\ldots(1+\alpha^1_{n+i-1})}, \qquad i = 2, 3, \ldots, N-1, \\ W^{(N)} &= \frac{\alpha^{N-1}_{n+1}}{(1+\alpha^1_n)\ldots(1+\alpha^1_{n+N-1})}. \end{aligned} \tag{7.6}$$

In the limit $N \to \infty$ one find the brackets between the fields $W^{(i)}$. Putting $W^{(1)} \equiv 1$, we have

$$\begin{aligned} \{W^{(p)}_n, W^{(q)}_{n+m}\} &= W^{(p)}_n W^{(q)}_{n+m}(1 - W^{(2)}_{n+m-1} - W^{(2)}_{n+p}) - W^{(q+m)}_n W^{(p-m)}_{n+m} \\ &\quad - W^{(p+1)}_n W^{(q)}_{n+m} + W^{(p)}_n W^{(q+1)}_{n+m-1} \quad \text{for } m \le p,\ \ q+m \ge p+1, \\ \{W^{(p)}_n, W^{(q)}_{n+m}\} &= W^{(p)}_n W^{(q)}_{n+m}(-W^{(2)}_{n+m-1} + W^{(2)}_{n+m+p}) \\ &\quad - W^{(p)}_n W^{(q+1)}_{n+m-1} - W^{(p)}_n W^{(q+1)}_{n+m} \quad \text{for } m \ge 1, \quad p \ge m+q+1. \\ \{W^{(p)}_n, W^{(q)}_{n+p+1}\} &= -W^{(p)}_n W^{(2)}_{n+p} W^{(q)}_{n+p+1} - W^{(p+q)}_n \\ &\quad + W^{(p+1)}_n W^{(q)}_{n+p+1} + W^{(p)}_n W^{(q+1)}_{n+p}, \\ \{W^{(p)}_n, W^{(q)}_{n+p-q}\} &= -W^{(p)}_n W^{(2)}_{n+p-q-1} W^{(q)}_{n+p-q} + W^{(p)}_n W^{(q+1)}_{n+p-q-1}, \quad \text{for } p \ge q+1, \\ \{W^{(p)}_n, W^{(q)}_n\} &= -W^{(p)}_n W^{(2)}_{n+p} W^{(q)}_n + W^{(p+1)}_n W^{(q)}_n \quad \text{for } q \ge p+1, \end{aligned} \tag{7.7}$$

Notice that the bracket (7.7) can be written in the form

$$\{\,,\,\}_{(7.7)} = -\{\,,\,\}_1 + \{\,,\,\}_2,$$

where $\{\,,\,\}_1$ is defined by Eq. (7.5). Concluding this section, we would like to highlight several points:

- Distinct from continuous case, for any *finite* N, algebra LW_N does not form a subalgebra of LW_∞. It forms only a *subspace*, defined by restriction $W^{(i)} = 0$ for $i \ge N$.
- In continuous case there exists the so-called *two-boson realization* of KP hierarchy [49], in which W_∞-algebra generators are expressed in terms of two $u(1)$ currents. Analogous construction happens to exist on the lattice. Fields forming Poisson algebra (7.7) can be realized in terms of two lattice $u(1)$ currents [8] $u_n = t_{2n}$ and $v_n = t_{2n+1}$, commuting as

$$\{t_n, t_{n+1}\} = -t_n t_{n+1}. \tag{7.8}$$

- Under properly defined continuous limit the brackets 1 and 2 become the corresponding Poisson structures of the KP hierarchy (resp. linear w_∞ and non-linear W_∞ algebras).

7.2. Integrable model associated with $LWA_{+\infty}$ algebra

Define the affine vertex of $\widehat{sl_N}$ as $a_n^0 = \prod_{i=1}^{N-1} (a_n^i)^{-1}$. The corresponding screening operator associated with the imaginary root of sl_N is

$$Q_0 = \sum_{n \in Z} a_n^0.$$

Differential $\hat{Q} = Q_0 + Q = \sum_{j=0}^{N} Q_j$ may be considered as the Hamiltonian of $\widehat{sl_N}$–Toda system. According to definitions of the work [30], space of integrals of motion of this system is defined as an intersection

$$Ker(ad_{Q_0}) \cap Ker(ad_{Q_1}) \cap \cdots \cap Ker(ad_{Q_{N-1}}) \cap \Pi_0/\partial\Pi_0. \tag{7.9}$$

The word *integrals* is contained in the last intersection because of obvious isomorphism

$$\Pi_0/\partial\Pi_0 \cong \Pi_0^{int} \leftarrow \Pi_0 : \sum_n .$$

Before describing the space (7.9), let us take a look at a simpler problem. It is almost a trivial statement, that a system associated with the pair of brackets $\{\,,\}_1$ and $\{\,,\}_2$ is integrable, with an infinite number of conservation laws in involution. One just has to have *two* integrals, commuting under both brackets. The simplest choice is [7]

$$I^{(1)} = \sum_n W_n^{(2)}, \qquad I^{(2)} = \sum \left(\frac{(W_n^{(2)})^2}{2} + W_n^{(2)} W_{n+1}^{(2)} - W_n^{(3)} \right).$$

The subsequent procedure is obvious: using the bi-Hamiltonian structure, one can easily obtain the whole series of conservation laws in involution by the recursive procedure. The answer for any N (essentially, including $N = \infty$) can be found in [8]. We rewrite it here for completeness. For a given N, the series is given by

$$I_N^{(k)} = \frac{1}{k} \mathrm{Tr}(\mathcal{L}_N^k), \tag{7.10}$$

where Lax matrix $\mathcal{L}_N$ is conveniently defined as

$$(\mathcal{L}_N)_{n,m}^{-1} = \delta_{n,m+1} - W_n^{(2)} \delta_{n,m} + W_n^{(3)} \delta_{n,m-1} - \cdots + (-1)^{N-1} W_n^{(N)} \delta_{n,m-N+1}. \tag{7.11}$$

Introducing the translation matrix $\Lambda_{n,m} = \delta_{n,m-1}$ and diagonal matrices $W_{n,m}^{(i)} = W_n^{(i)} \delta_{n,m}$ we can write in compact notations

$$\mathcal{L}_N = \Lambda^{-1} \cdot \frac{1}{1 - L \cdot \Lambda + W^{(3)} \cdot \Lambda^2 - \cdots + (-1)^{N-1} W^{(N)} \Lambda^{N-1}}. \tag{7.12}$$

It really is the L-operator of our dynamical system, because the evolution equations can be written in the form

$$\frac{\partial \mathcal{L}_N}{\partial t_p} = [A_N^{(p)}, \mathcal{L}_N], \tag{7.13}$$

where $A_N^{(p)} = (\mathcal{L}_N^p)_+$. Now let us return to our original problem of description of the space (7.9). Obviously, all of the integrals above commute with $Q = \sum_{j=0}^{N} Q_j$ by construction (7.3). In addition, direct calculation shows that *they also commute with* Q_0. Notice that of N vertex operators corresponding to $\widehat{sl_N}$ we needed only $N-1$ corresponding to simple roots to construct LWA_N generators and integrals of motion. In principle, we could pick up *any* $N-1$ vertex operators, and follow the same steps. Thus, the space of integrals of motion for $\widehat{sl_N}$ lattice Toda system can be described in terms of generating functions as

$$R_{\widehat{SL_N}}(\lambda_1, \ldots, \lambda_N) = \sum_{i=1}^{N} R_{SL_N}^{(i)}(\lambda_i), \tag{7.14}$$

where $R_{SL_N}^{(i)}(\lambda_i) = \sum_{s=0}^{\infty} I^{(s)} \lambda_i^s$ is the generating function for the conservation laws of the lattice N-KdV hierarchy, associated with the roots
$\{\alpha_1, \alpha_2, \ldots, \widehat{\alpha_i}, \ldots, \alpha_N\}$.

7.3. Embedding of the Lattice NLS into the lattice KP hierarchy

One can prove by direct calculation that evolution of the fields

$$M_n^p = \frac{e_n f_{n+p}}{h_n h_{n+1} \ldots h_{n+p-1}} \tag{7.15}$$

is consistent with the lattice KP hierarchy (7.13) under the identification

$$(\mathcal{L}_\infty)_{n,n+p} = (-1)^p M_n^{p+1}.$$

Notice, however, that as defined by Eq. (7.15), variables $\{M_n^p\}_{p=1}^{\infty}$ are not functionally independent. There is a set of quadratic relations, such as, e.g. $M_n^2 M_{n+1}^2 = M_n^3 M_{n+1}^1$, which may be interpreted as Plucker relations of some Grassmanian. Two independent generators of the whole family are M_n^0 and M_n^1, encountered earlier in Section 5. It is easy to check that they as well form FTV algebra, under the identification

$$A_{2n} = M_n^0, \qquad A_{2n+1} = M_n^1.$$

Thus, one may view the the embedding of the lattice NLS into the lattice KP as a *non-Abelian* two field realization of lattice KP.

8. Concluding remarks

In this paper we have studied the lattice analogues of various conformal theories as well as their integrable perturbations. We have found that when described in proper invariant terms, many of the well-known continuous constructions have their match on the lattice. We have explicitly described for the first time lattice analogues of the DS reduction and of the Sugawara construction. In the framework of the lattice WZW the lattice Sugawara energy–momentum tensor has been constructed. We have described lattice Maxwell–Bloch system as a "chiral perturbation" of the lattice WZW model by the field of spin one. Evolution equations under the integrals of motion of this system have been found to form an integrable hierarchy, which is naturally perceived as a lattice analogue of the NLS hierarchy. Finally, we have found an embedding of this lattice NLS hierarchy into the lattice KP hierarchy, again in complete analogy with the continuous case.

One of the problem that remained open is giving a geometrical description of the lattice MB system using the Lie group cosets, in analogy with continuous case [4].

We have described the spaces of IMs for several integrable systems, using Lax representation and bi-Hamiltonian structure. It would be extremely interesting to compare the results of cohomological [26] and St.-Petersburg [22,33] calculations with our answers. Recently Kryukov calculated the first three integrals in the quasi-classical limit of lattice sine-Gordon theory [39], using the generating function from the paper of Faddeev and Volkov [22]. After careful comparison, we found that his integrals of motion can be expressed in terms of certain linear combinations of ours.

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